سكشن رقم ٧

ازاي مسائل ال Quiz

A) Prove that $\frac{d}{dz}(z^2 \bar{z})$ doesn't exist any where

B) show that $\frac{d}{dz}(Ln(z)) = \frac{1}{z}$

Complex integration

$$\int_a^b f(z)\,dz = \int_c f(z)\,dz \longrightarrow$$ تكامل على مسار غير منتظم

$$\oint_c f(z)\,dz \longrightarrow$$ تكامل على مسار مغلق

$\boxed{I}$ $\int_c f(z)\,dz = ?$

$f(z) = u + iv \quad ; \quad z = x + iy$

~~dz = dx~~ $\qquad dz = dx + i\,dy$

$$\therefore \int_c f(z)\,dz = \int_c (u + iv)(dx + i\,dy)$$

$$= \int (u\,dx - v\,dy) + i(v\,dx + u\,dy)$$

$\boxed{1}$ Sec 7

① ثم نجعله كله في x أو dx . أو y ، dy أو دور بارامترية

① <u>Circle</u> $(x-x_0)^2 + (y-y_0)^2 = a^2$

$x = x_0 + a\cos t \quad ; \quad y = y_0 + a\sin t$

$dx = -a\sin t\,dt \quad ; \quad dy = a\cos t\,dt$

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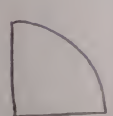
② <u>قطع ناقص</u>

$$\frac{(x-x_0)^2}{a^2} + \frac{(y-y_0)^2}{b^2} = 1$$

$x = x_0 + a\cos t \qquad y = y_0 + b\sin t$

$dx = -a\sin t\,dt \qquad dy = b\cos t\,dt$

الفترات ⇐



$0 \leq t \leq \frac{\pi}{2}$

$0 \leq t \leq \pi$

$0 \leq t \leq 2\pi$

$\boxed{3}$ <u>قطع زائد</u>

$$\frac{(x-x_0)^2}{a^2} - \frac{(y-y_0)^2}{b^2} = 1$$

$x = x_0 + a\cosh t \qquad y = y_0 + b\sinh t$

$dx = a\sinh(t)\,dt \qquad dy = b\cosh(t)\,dt$

٤ قطع مكافئ :

④ $y = a x^2$

$x = t$ $\qquad y = a t^2$

$dx = dt$ $\qquad dy = 2at\,dt$

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⑤ $|z - z_0| = a$

$x = x_0 + a\cos\theta$ $\qquad y = y_0 + a\sin\theta$

$dx = -a\sin(\theta)\,d\theta$ $\qquad dy = a\cos\theta\,d\theta$

$0 \leq \theta \leq \frac{\pi}{2}$

$0 \leq \theta \leq \pi$

$0 \leq \theta \leq 2\pi$

$\boxed{1}$ Evaluate $\int_C f(z)\,dz$ from $0+i$ to $2+5i$

where $f(z) = (3x+y) + i(x-2y)$

a) along $y = x^2 + 1$

b) along line $(0,1)$ to $(0,5)$

$\boxed{\text{solution}}$

$f(z) = (3x+y) + i(x-2y)$; $dz = dx + idy$

a) $y = x^2 + 1$

$f(z) = (3x + x^2 + 1) + i(x - 2x^2 - 2)$

$dz = dx + idy$

$= dx + i2x\,dx$

$$\therefore \int_c f(z)dz = \int_0^2 \left[(3x+x^2+1) + i(x-2x^2-2)\right](dx + i2x\,d\cancel{y})$$

$$= \int_0^2 (3x + x^2 + 1)dx + i(x - 2x^2 - 2)dx +$$

$$i(3x + x^2 + 1)2x\,dx - (x - 2x^2 - 2)2x\,dx$$

$$= \int_0^2 (3x + x^2 + 1 - 2x^2 + 4x^3 + 4x)\,dx$$

$$+ i(x - 2x^2 - 2 + 6x^2 + 2x^3)\,dx$$

$$= \int_0^2 (7x - x^2 + 1 + 4x^3)dx + i(3x + 4x^2 - 2 - 2x^3)\,dx$$

$$= \left(\frac{7x^2}{2} - \frac{x^3}{3} + x + x^4\right)\Big|_0^2 + i\left(\frac{3x^2}{2} + \frac{4x^3}{3} - 2x + \frac{x^4}{2}\right)\Big|_0^2$$

$$= \frac{88}{3} + i\,\frac{62}{3}$$

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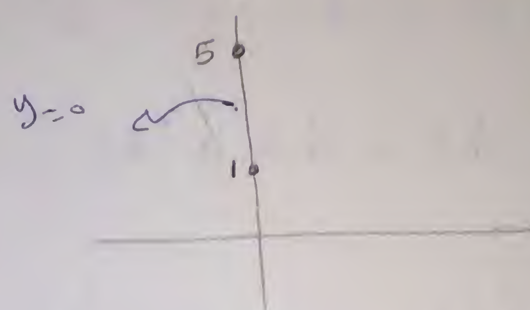
b)



$x = 0 \qquad dx = 0$

$f(z) = y - i2y$

$dz = i\,dy$

$$\int_C f(z)\,dz = \int_1^5 (y - i2y)\,i\,dy$$

$$= \int_1^5 iy\,dy + 2y\,dy$$

$$= i\,\frac{y^2}{2}\Big|_1^5 + y^2\Big|_1^5$$

$$\int_1^5 f(z)\,dz = 24 + i12$$

* evaluate $\int_C \bar{z}\,dz$ for $x = 2\cos t$, $y = \sin t$

from $t = 0$ to $t = \frac{\pi}{2}$

$$\bar{z} = x - iy = 2\cos t - i\sin t$$

$$dz = dx + i\,dy = -2\sin(t)\,dt + i\cos(t)\,dt$$

$$\int_0^{\frac{\pi}{2}} (2\cos t - i\sin t)\,(-2\sin(t)\,dt + i\cos(t)\,dt)$$

$$= \int_0^{\frac{\pi}{2}} -4\cos t\sin t\,dt + i2\sin^2 t\,dt + i2\cos^2 t\,dt$$

$$+ \sin t\cos t\,dt$$

$$= \int_0^{\frac{\pi}{2}} -3\cos t\sin t\,dt + i2\,dt$$

$$= \int_0^{\frac{\pi}{2}} -3\,\frac{\sin 2t}{2} + 2i\,dt = \frac{3}{4}\cos 2t + i2t \Big|_0^{\frac{\pi}{2}}$$

$$= \frac{-3}{4} + \pi i$$

٢ تكامل على مسار مغلق

a) $\oint_c f(z)\,dz = 0$

← إذا كانت الدالة $f(z)$ ليس لها مقام ولا تحتوى على $\bar{z}$
أو $|z|$ أو أصفار مقام لا تقع داخل المنحنى (analytic)

b) $\oint_c \frac{f(z)}{z-a} = 2\pi i\, f(a)$

c) $\oint_c \frac{f(z)}{(z-a)^{n+1}} = \frac{2\pi i}{n!} \left.\frac{d^n f(z)}{dz}\right|_{z=a}$

← إذا كانت الدالة تحتوى على مقام واحد وصفره داخل
المنحنى c.

الأفكار:-

a) تعويض مباشر.

b) فيه أكثر من قوس تحت وواحد فيهم يقع داخل المنحنى والباقى لا.

c) أكثر من قوس تحت وأصفاره تقع داخل المنحنى:

$$\oint_c = \oint_{c_1} + \oint_{c_2}$$

Evaluate

$\boxed{1}$ $\oint_C e^{10z} \, dz$ b) $\oint_C \frac{z^2}{z-3} \, dz$ $|z| = 1$

$\boxed{c}$ $\oint_C \operatorname{sech}(z) \, dz$ $\boxed{d}$ $\oint_C z \cdot e^{-z} \, dz$

$\boxed{e}$ $\oint_C \frac{1}{z^2+2z+1} \, dz$ $\boxed{f}$ $\oint_C \frac{2z-1}{z^2-z} \, dz$

$|z| = \frac{1}{2}$

$\boxed{\text{sol}}$

$\boxed{1}$ $e^{10z} \, dz = 0$

$|z| = 1$

$\boxed{2}$ $\oint_C \frac{z^2}{z-3} \, dz = 0$

$\boxed{3}$ $\oint_C \operatorname{sech}(z) \, dz$.

$\text{Cosh} = \frac{e^{z} - e^{-z}}{2}$

$$\oint_C \operatorname{sech}(z) \, dz = \frac{2}{e^{z} - e^{-z}} = 0$$

$\boxed{8}$ sec 7

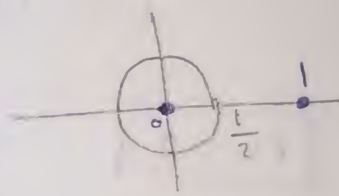
$\boxed{4}$ $\oint_c z \cdot e^{-z}\, dz = 0$

$\boxed{5}$ $\oint \frac{1}{z^2+2z+1}\, dz = 0$

$|z| = 1 \quad ; \quad z = -1 \pm i$

$\boxed{6}$ $\oint \frac{2z-1}{z^2-z}\, dz = \oint \frac{2z-1}{z(z-1)}\, dz$

$|z| = \frac{1}{2}$



$= \oint \frac{\overbrace{\frac{2z-1}{z-1}}^{f(z)}}{z} = 2\pi i \; f(a)$

$= 2\pi i \; f(0) = \boxed{2\pi i}$

$\boxed{7}$ $\oint_c \frac{7z-6}{z^2-2z} \quad ; |z| = 6$

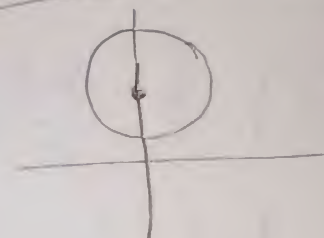
$= \oint \frac{7z-6}{z(z-2)}$ النقطتين داخل المنحنى.

$$= \int_{C_1} \frac{\frac{7z-6}{z-2}}{z} + \int_{C_2} \frac{\frac{7z-6}{z}}{z-2}$$

$$= 2\pi i\, f(0) + 2\pi i\, f(2)$$

$$= 6\pi i + 8\pi i = 14\pi i$$

$\boxed{8}$ $\int \frac{e^{2z^2}}{(z-i)^3}$ $|z-i| = 1$



$\boxed{n=2}$ Cause $n+1 = 3$

$\cancel{\oint = \frac{2\pi i}{2!}}$ $f(z) = e^{2z^2}$

$f'(z) = 4z\, e^{2z^2}$, $f''(z) = 4e^{2z^2} + 16z^2 e^{2z^2}$

$f''(i) = 4e^{-2} + 16(-1)e^{-2z}$

$= -12 e^{-2z}$

$\cancel{\oint = \frac{2\pi i}{2!}}$

$$\oint = \frac{2\pi i}{2!}\left(-12 e^{-2z}\right) = -12 e^{-2} \pi i$$

$\boxed{10}$ sec 7